Optimal Portfolios in Good Times and Bad

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Recent experience with emerging market investments and hedge funds has highlighted the fact that risk parameters are unstable. To address this problem, we introduce a procedure for identifying multivariate outliers and use the outliers to estimate a new covariance matrix. We suggest that a covariance matrix estimated from outliers characterizes a portfolio's riskiness during market turbulence better than a full-sample covariance matrix. We also introduce a procedure for blending an inside-sample covariance matrix with one from an outlier sample. This procedure enables one to express views about the likelihood of each risk regime and to differentiate one's aversion to them. Our framework collapses to the Markowitz mean–variance model if (1) we set the probabilities of the inside and outlying covariance matrixes equal to their empirical frequencies, (2) we are equally averse to both risk regimes, and (3) we estimate the inside and outlying covariances around the full sample's mean.

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Markowitz (1952) introduced an efficient process for selecting portfolios. His landmark innovation, mean–variance optimization, requires financial analysts to estimate expected returns, standard deviations, and correlations.[1] Markowitz showed how analysts could use this information to combine assets optimally so that for a particular level of expected return, the resulting portfolio would offer the lowest possible level of expected risk, usually measured as standard deviation or variance. A continuum of these portfolios displayed in dimensions of expected return and standard deviation is the efficient frontier.

The implementation of portfolio theory introduces several problems, however, including the estimation of the requisite parameters and the sensitivity of the resulting portfolios to small differences in those parameters. Often, the parameters are unreliable because they are estimated from small samples.[2] We address a problem that applies to large samples as well as small samples.

Recent experience with failed hedge funds has focused attention on a serious limitation of the typical risk-estimation procedure, which is to weight a sample's observations equally in order to estimate risk parameters. Although this procedure may produce reasonable estimates for the full investment horizon, it probably misrepresents a portfolio's risk attributes during periods of turbulence or financial crisis. In turbulent markets, asset returns tend to become more volatile and more highly correlated. Thus, the diversification that characterizes the sample on average disappears when it is most needed.

For example, Siconolfi, Raghavan, and Pacelle (1998) wrote in the *Wall Street Journal*:

> In mid-August, Russia abruptly defaulted on part of its debt and let the ruble fall, triggering a flight by investors from all types of risk into safe investments. That devastated some of LTCM's [Long-Term Capital Management's] bets, leading to the huge losses of Aug. 21. (pp. A18–A19)

Another way to think about this issue is to distinguish time-measured observations from event-measured observations. There is a certain arbitrariness to measuring returns simply as a function of units of time. In some periods, no significant events will take place to cause prices to change, so returns will essentially reflect noise. In other periods, several important events will influence returns. But the typical estimation of risk parameters assigns as much importance to the periods with no significant events as it does to the event-filled periods. A more informative alternative might be to calibrate returns as a function of events and then to estimate risk parameters from

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these event-measured observations. This approach might provide a better representation of a portfolio's likely performance during turbulent markets than the time-measured approach.

To address the issue that risk parameters are unstable, we introduce two innovations to portfolio optimization. First, we present a procedure for estimating risk parameters from multivariate outliers. This approach is based on the rationale that outliers are more likely to be associated with stress-related events than with noise. Keep in mind, however, that we define stress as periods that are unusual, not necessarily periods characterized only by low or negative returns.

Second, we show how to construct portfolios that simultaneously balance risk parameters estimated from "quiet," or low-event, times with those estimated from outlier observations representing turbulent, stressful times. This innovation, which was suggested to us by Markowitz, is an adaptation of Chow's (1995) optimization algorithm for combining absolute and relative performance.

In addition, we provide empirical results to demonstrate both procedures.

Multivariate Outliers

An outlier in a return series for a single asset is straightforward to identify. It is simply a return that falls outside a chosen confidence interval around the expected return. For example, we could define an outlier as a return that falls within the tails that comprise 25 percent of the distribution, 12.5 percent on either side. Thus, with μ the expected continuous return and σ the standard deviation of continuous returns, if continuous returns are normally distributed, an outlier for a single return series is any continuous return that is greater than $\mu + 1.175\sigma$ or less than $\mu - 1.175\sigma$ because for a normal distribution, 75 percent of the returns are likely to fall within 1.175σ of the expected return μ.

A multivariate outlier is more difficult to identify. It represents a set of contemporaneous returns that is collectively unusual for one or more reasons. One of the returns may be sufficiently far from its mean to qualify the collection of returns for that period as an outlier, or a pair of returns that are highly correlated may exhibit a sufficient difference in their returns to render the period unusual. Thus, a multivariate outlier may result from the unusual performance of an individual asset or from the unusual interaction of a combination of assets, none of which are necessarily unusual in isolation.

How does one determine explicitly whether to classify an observation as "usual" or as an "outlier"? **Figure 1** presents a scatterplot of two independent return series (for Asset 1 and for Asset 2) with equal variances. To identify outliers, we first draw a circle around the mean of the data. The radius we choose for the circle is our tolerance for outliers and takes into account the variances of the two return series. As shown in **Figure 2**, the inside circle is our boundary for defining outliers. To determine which observations are outliers, we calculate the equation of a circle for each observation with its center located at the mean of the data and its perimeter passing through a given observation. If the radius of this circle is greater than our "tolerance radius," we define that observation as an outlier. In Figure 2, for example, the observation designated by the star is an outlier because the radius of the circle passing through the star is greater than the radius of our tolerance circle.

Figure 1. Scatterplot of Independent Return Series with Equal Variances

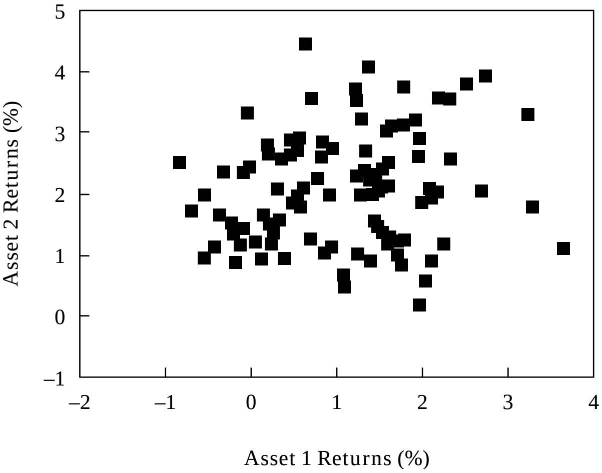
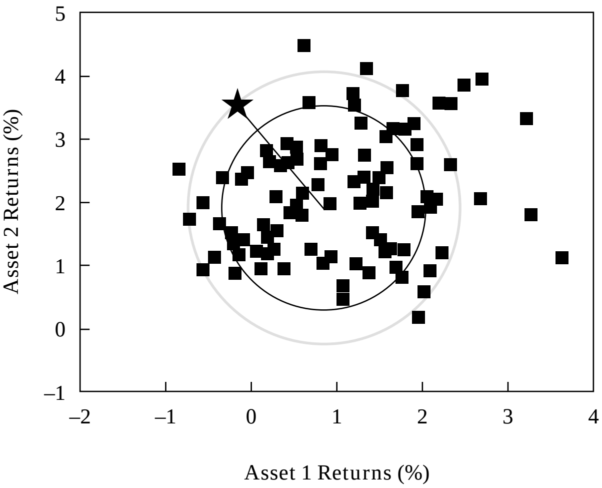


Figure 2. Tolerance Circle and One Outlier Circle



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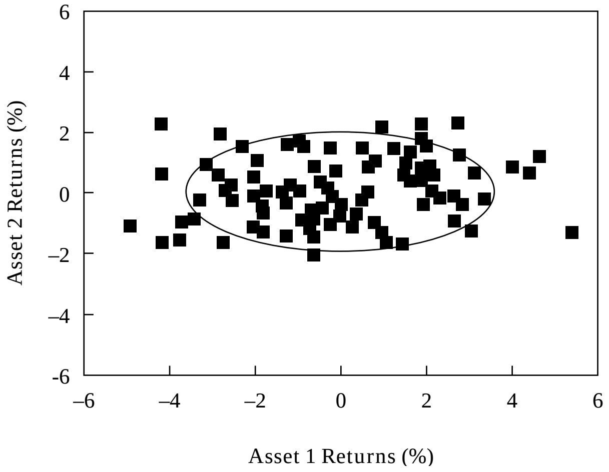
This approach is appropriate for a sample of returns if the returns are uncorrelated and have the same variance. When the return series have different variances, a circle is no longer appropriate for identifying outliers. The scatterplot in **Figure 3** is for two uncorrelated return series that have unequal variances and shows that under these conditions, an ellipse is the appropriate shape for defining the outlier boundary.[3] As for the circle, we start with our "tolerance ellipse," and for each point, we calculate an ellipse with a parallel perimeter. Then, we compare their boundaries.

Figure 3. Scatterplot of Uncorrelated Asset Returns with Unequal Variances



In **Figure 4**, we relax the remaining assumption and allow for a nonzero correlation. That the ellipse is positively sloped implies that the return series are positively correlated. With correlated returns, we generate tolerance ellipses whose axes are rotated (shown by the straight lines in Figure 4). Our basic intuition for identifying an outlier remains unchanged, but when the return series are correlated or when the sample includes more than three return series, we must use matrix algebra for the exact computation of an outlier. This procedure is described in Appendix A.

Optimal Portfolios with Event-Varying Covariance Matrixes

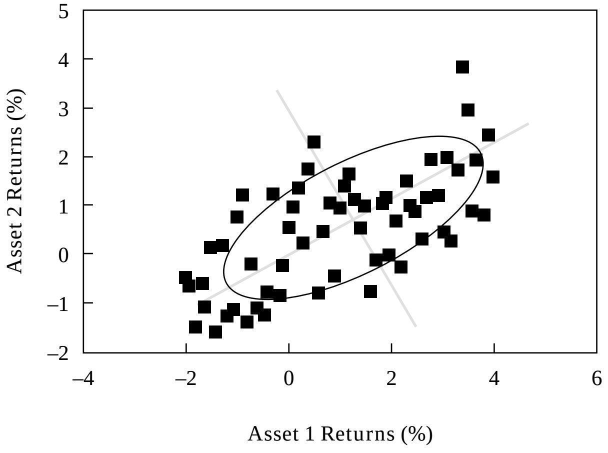
In the previous section, we showed how to identify multivariate outliers from which to estimate risk parameters. These observations are essentially representative of turbulent markets, which are characterized by higher-than-normal volatility and correlations. One might argue that investors should focus only on long-term performance. We suggest, however, that their portfolios may not survive to generate long-term performance if the portfolios cannot withstand exceptional periods of market turbulence.

Even so, we do not advocate the use of a covariance matrix estimated only from outlying observations. To focus on stressful periods alone would be shortsighted and could lead investors to hold unduly conservative portfolios that would fail to achieve their long-term objectives.

We argue, instead, that investors care about risk during quiet periods and risk during stressful periods. We address this dual focus in two ways. First, we allow investors to estimate two separate covariance matrixes—one from the inside observations to represent a quiet risk regime and the other from the outlier observations to represent a stressful risk regime.[4] We then assign a probability of occurrence to each risk regime. Second, we allow investors to specify different degrees of risk aversion toward the two regimes. Investors can, of course, express a view about the relative likelihood of the two regimes and at the same time assign different risk aversions to them, but separating the two parameters is important: One is a forecast, whereas the other is a behavioral parameter. Too often, a forecast of risk is confused with an attitude toward risk.

Once we have estimated the quiet and stressful covariance matrixes, we blend them into a single covariance matrix that can reflect one's view about the likelihood of each regime and one's attitude toward each regime. We then incorporate this blended covariance matrix into the standard optimization algorithm for selecting portfolios. This procedure is described in Appendix B.

Figure 4. Scatterplot of Correlated Asset Returns with Unequal Variances



Our approach for selecting portfolios based on a blended covariance matrix is internally consistent in the following sense: The blended covariance matrix equals the full-sample covariance matrix if (1) we set the probabilities of the inside and outlying covariance matrixes equal to their empirical frequencies, (2) we are equally averse to both risk regimes, and (3) we estimate the inside and outlying covariances around the full-sample mean. Thus, our framework nests the original Markowitz mean–variance model in an intuitive manner.

Empirical Results. We applied our technique to a sample of actual returns for eight asset classes beginning in January 1988 and continuing through September 1998. These return series are for eight of the asset-class benchmarks used by the Harvard Management Company to evaluate the policy portfolio of Harvard's endowment fund. They are domestic equities (in this case, "domestic" means "U.S."), domestic bonds, foreign (non-U.S., developed market) equities, foreign bonds, high-yield U.S. bonds, emerging market equities, commodities, and cash equivalents.

Table 1 shows the returns of these eight asset classes for the first nine months of 1998. August and September 1998, shaded on the table, qualify as outliers based on an outer boundary for which 25 percent of the multivariate distribution is excluded. August and September 1998 are 2 of 27 months of the 129 months in the full sample that qualified as outliers, which equals 20.9 percent of the months. That fewer than 25 percent of the months were selected implies that the multivariate distribution of these returns is slightly plattykurtic; that is, more of the observations are clustered near the mean than theory would predict.

The equity returns in August 1998 were extraordinarily low. It was the month in which Russia defaulted on its sovereign debt, which triggered a flight from risky assets. Even commodities, which typically diversify financial assets, generated significant losses in August. The returns in September 1998 were not unusual in their magnitude, but foreign equities, which typically move in tandem with domestic and emerging market equities, generated a loss whereas domestic and emerging market equities produced significant gains. Commodities also generated a gain during this period. February produced high returns, but according to our outlier definition, those results were not strange.

When we consider more than three asset classes, why a particular set of returns qualifies or fails to qualify as an outlier is not always clear because, as mentioned earlier, we cannot visualize the observations that lie outside the boundary. We must rely on mathematical techniques to identify the higher-dimension outliers.

Table 2 provides the annualized standard deviations and correlations estimated from the full sample of 129 months. The average standard deviation of these asset classes based on the full sample is 11.67 percent, and the simple average correlation is approximately 0.12.

Table 3 presents the same information for the sample of 27 outliers. The average standard deviation for the outlier sample is 18.27 percent, a 57 percent increase over the full-sample standard deviation. The average correlation is also higher for the outlier sample, but at approximately 0.14, it is only 15 percent higher than for the full sample.

The change in the average correlation obscures important details, however, about the diversification properties of the various asset classes. For example, as **Table 4** shows, if commodities are excluded, the average correlation for the other asset classes in the full sample is approximately 0.17 but for the outlier sample, the average correlation increases 36 percent, from 0.17 to approximately 0.23. In contrast, the average correlation of commodities with all the other asset classes declines in the outlier sample from the full

Table 1. Returns for Eight Asset Classes, January 1998–September 1998
(25 percent boundary)

Month/ Day of 1998	Domestic Equities	Foreign Equities	Emerging Market Equities	Domestic Bonds	Foreign Bonds	High-Yield Bonds	Commodities	Cash
1/31	0.59%	4.29%	–5.70%	1.99%	0.63%	1.21%	–1.61%	0.47%
2/28	7.35	6.88	10.30	–0.62	1.44	1.49	–6.60	0.47
3/31	5.01	3.77	3.22	0.20	–1.41	–1.08	–0.52	0.47
4/30	1.14	0.87	–0.80	0.40	2.24	0.71	–2.73	0.47
5/31	–2.26	–0.17	–11.91	1.59	0.03	2.09	–4.72	0.47
6/30	3.45	0.73	–10.30	1.85	–0.30	0.55	–3.55	0.47
7/31	–1.66	1.17	2.87	–0.18	0.33	–0.41	–7.86	0.47
8/31	–15.15	–12.43	–25.42	4.06	2.73	–11.80	–5.90	0.47
9/30	6.86	–3.03	4.89	3.57	6.67	–0.87	10.22	0.46

Table 2. Full-Sample Risk Parameters

	Domestic Equities	Foreign Equities	Emerging Market Equities	Domestic Bonds	Foreign Bonds	High-Yield Bonds	Commodities	Cash
A. Standard deviation	12.99%	17.04%	22.62%	6.79%	9.57%	8.00%	15.94%	0.44%
B. Correlation								
Domestic equities	1.00	0.50	0.39	0.39	0.05	0.51	–0.09	0.06
Foreign equities		1.00	0.37	0.17	0.48	0.33	–0.09	–0.07
Emerging market equities			1.00	–0.08	–0.14	0.34	–0.05	–0.06
Domestic bonds				1.00	0.28	0.13	–0.08	0.10
Foreign bonds					1.00	–0.03	0.03	–0.04
High-yield bonds						1.00	–0.14	–0.13
Commodities							1.00	0.21
Cash								1.00

Table 3. Outlier-Sample Risk Parameters

	Domestic Equities	Foreign Equities	Emerging Market Equities	Domestic Bonds	Foreign Bonds	High-Yield Bonds	Commodities	Cash
A. Standard deviation	20.55%	27.35%	34.29%	9.74%	14.09%	14.85%	24.83%	0.47%
B. Correlation								
Domestic equities	1.00	0.57	0.49	0.43	0.14	0.73	–0.23	0.00
Foreign equities		1.00	0.40	0.27	0.49	0.42	–0.29	0.00
Emerging market equities			1.00	0.08	–0.18	0.56	–0.25	–0.20
Domestic bonds				1.00	0.32	0.13	–0.14	0.20
Foreign bonds					1.00	–0.12	–0.09	0.12
High-yield bonds						1.00	–0.18	–0.01
Commodities							1.00	0.20
Cash								1.00

sample by a factor of nearly 5, from –0.03 to approximately –0.14. Hence, the full-sample correlations belie the weaker diversification properties of financial assets in times of stress but understate the diversification benefits of commodities when markets experience turbulence.

Note that we used 25 percent to define the outlying area of the distribution. One might be tempted to explore more extreme outliers, which would reduce the number of outliers, but this reduction could cause serious problems in the estimation of the covariance matrix. In order for the covariance matrix estimator to be reasonably precise, observations should always far outnumber assets. As a rule, observations should be at least twice the number of assets.[5] In our example, we identified 27 observations for an eight-by-eight covariance matrix, which produced 36 parameters.

Optimal Portfolios. The composition of the optimal portfolio shifts when we switch from using the full-sample risk parameters to using the outlier-sample risk parameters. For both optimizations, we assumed the following mean returns for the asset classes:[6]

Domestic equities	10.00%
Foreign equities	10.25
Emerging market equities	12.00
Domestic bonds	7.00
Foreign bonds	7.25
High-yield bonds	8.00
Commodities	6.00
Cash equivalents	5.00

Because the mean returns were preset, the changes when different risk parameters were used reflect only the differences in those parameters.

Table 5 shows the optimal portfolio weights based on the full-sample risk parameters and the outlier-sample risk parameters under the assumption that we are willing to sacrifice 2.5 units of expected return to lower our portfolio's variance by

Table 4. Full-Sample and Outlier-Sample Differences (averages)

Sample	Standard Deviation	Correlation	Correlation of All Assets Excluding Commodities	Correlation of Commodities with All Other Assets
Full	11.67%	0.12	0.17	–0.03
Outlier	18.27	0.14	0.23	–0.14

Table 5. Asset Mix for Full-Sample and Outlier-Sample Optimal Portfolios

Asset Class	Full-Sample Optimal Mix	Outlier-Sample Optimal Mix
Domestic equities	25%	2%
Foreign equities	0	0
Emerging market equities	16	11
Domestic bonds	8	28
Foreign bonds	26	25
High-yield bonds	22	16
Commodities	3	12
Cash	0	7

1 unit. Given that degree of risk aversion, together with the covariance matrix estimated from the full sample of monthly returns, the full-sample optimal portfolio would contain primarily domestic equities, foreign assets, and high-yield bonds. Very little of the portfolio would be allocated to domestic bonds and commodities. **Table 6** provides the portfolios' expected returns and standard deviations in normal and in stressful times. This allocation, assuming the full-sample covariance matrix characterizes the risk of our investment horizon, offers an expected return of 8.82 percent with a standard deviation of only 7.27 percent. In an environment that is better represented by the risk parameters associated with the outlier sample, however, this portfolio will experience an almost 70 percent increase in standard deviation.

Table 6. Full-Sample and Outlier-Sample Optimal Portfolios

Statistic by Environment	Full-Sample Optimal Mix	Outlier-Sample Optimal Mix
Normal environment		
Expected return	8.82%	7.58%
Standard deviation	7.27	4.37
Stressful environment		
Expected return	8.82	7.58
Standard deviation	12.32	7.20

Table 5 shows that if we were structuring an optimal portfolio based on the outlier-sample covariance matrix, we would reduce the equity component to 13 percent, increase commodities to 12 percent, and increase fixed-income investments to about 75 percent of the portfolio. This portfolio would be optimal for periods that are characterized by the covariance matrix estimated from the sample's outliers as long as our risk aversion remained constant. In turbulent times, as Table 6 shows, this portfolio's volatility would be 7.20 percent, in contrast to the volatility of the portfolio based on the full sample (12.32 percent). The outlier-sample optimal mix also has lower volatility than the full-sample optimal mix in a normal environment, but the shift in assets is not without cost. The changes in allocation reduce expected return from 8.82 percent to 7.58 percent.

Herein lies the problem. If we optimize based on the full-sample covariance matrix, the portfolio will be significantly suboptimal in a period of financial stress and, indeed, may not survive such a period without unpropitious adjustments. If we optimize based on the outlier covariance matrix, the portfolio's expected return for the full horizon will be lower than desired. What to do?

As with many choices, the best solution is to compromise. **Table 7** presents optimal portfolios based on blended covariance matrixes, their expected performance during quiet environments (times characterized by the inside sample only), normal environments (full-sample times), and stressful environments (outlier-sample times).

The first portfolio shown in Table 7 is the optimal portfolio based on the full-sample covariance matrix. The next portfolio assumes that we are 1.5 times as averse to outlier risk as we are to risk during quiet periods and that outliers will occur with the same frequency as they occurred empirically in our study (20.9 percent of the months). As expected, this relatively higher aversion to outlier risk shifts the full-sample optimal portfolio away from equities and toward commodities and domestic bonds.

The next column shows the optimal portfolio if we are equally averse to outlier risk and quiet-period risk and we believe turbulent periods will occur 50 percent of the time rather than their empirical frequency. Again, the optimal portfolio under these conditions is more conservative than the full-sample portfolio. It is more conservative because, even though we are equally averse to both environments, we expect turbulent periods to occur more frequently than they did empirically.

The final column shows the optimal portfolio if we assume that outlier events will occur 50 percent of the time, rather than their empirical frequency, and we are 1.5 times as averse to outlier risk as we are to risk during quiet periods. These assumptions emphasize the outlier sample in two ways—by assigning it a greater probability than its empirical frequency and by raising our relative aversion to outlier risk. This dual emphasis results in an optimal portfolio that, of the choices in Table 7, most closely resembles the portfolio estimated solely from the outlier sample.

Conclusion

We introduced a methodology to address the instability of risk parameters. Specifically, we identified

Table 7. Optimal Portfolios from Blended Covariance Matrixes

	Full-Sample Optimal Mix	Empirical Probability / Higher Outlier Aversion	Equal Probability / Equal Aversion	Equal Probability / Higher Outlier Aversion
Domestic equities	25%	21%	12%	9%
Foreign equities	0	0	0	0
Emerging market equities	16	16	14	13
Domestic bonds	8	14	23	25
Foreign bonds	26	26	26	27
High-yield bonds	22	17	14	15
Commodities	3	6	10	11
Cash	0	0	0	0
Quiet environment				
Expected return	8.82	8.60	8.18	8.05
Standard deviation	5.14	4.91	4.50	4.38
Normal environment				
Expected return	8.82	8.60	8.18	8.05
Standard deviation	7.27	6.70	5.81	5.58
Stressful environment				
Expected return	8.82	8.60	8.18	8.05
Standard deviation	12.32	11.03	9.13	8.64

multivariate outliers and used these outliers to estimate a new covariance matrix. We believe that a covariance matrix estimated from outliers provides a better representation of a portfolio's riskiness during periods of market turbulence than does a covariance matrix estimated from the full sample of observations.

We introduced a procedure for blending an inside-sample covariance matrix with an outlier-sample covariance matrix. The procedure enables investors to express their views about the likelihood of each risk regime and to differentiate their aversion to the regimes.

Our empirical results supported the view that volatility and correlations estimated from outliers differ significantly from full-sample estimates. In addition, we identified optimal portfolios from both covariance matrixes while holding expected returns and risk aversion constant. Given our sample, the volatility of the optimal portfolio estimated from the full-sample covariance matrix nearly doubled when the portfolio was subjected to the outlier-sample covariance matrix. As expected, the outlier-sample covariance matrix produced a much more conservative optimal mix than the full-sample matrix but with a concomitantly lower expected return.

Results based on covariance matrixes that were blended from quiet and turbulent regimes showed the sensitivity of portfolio weights to variations in one's views about the relative likelihood of quiet and turbulent periods and one's relative aversion to each risk regime.

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Appendix A. Identification of Multivariate Outliers

The calculation of a multivariate outlier is given by the following equation:

$$d_t = (y_t - \mu)\,\Sigma^{-1}(y_t - \mu)', \tag{A1}$$

where

d_t = vector distance from multivariate average
y_t = return series
μ = mean vector of return series y_t
Σ = covariance matrix of return series y_t

We assume the return series y_t is normally distributed with a mean vector μ and a covariance matrix Σ. For 12 return series, for example, an individual observation of y_t would be the set of the 12 asset returns for a specific measurement interval. We choose a tolerance "distance" and examine the distance, d_t, for each vector in the series. If the observed d_t is greater than the tolerance distance, we define that vector as an outlier.

For two uncorrelated return series, Equation A1 simplifies to the following equation:

$$d_t = \frac{(y - \mu_y)^2}{\sigma_y^2} + \frac{(x - \mu_x)^2}{\sigma_x^2}, \tag{A2}$$

which is the equation of an ellipse with horizontal and vertical axes.

If the variances of the return series are equal, Equation A2 simplifies to a circle.

For the general n-return normal series case, d_t is distributed as a chi-square distribution with n degrees of freedom. Under this assumption, if an outlier is defined as falling beyond the outer 25 percent of the distribution and we have 12 return series, our tolerance boundary is a chi-square score of 14.84. Using Equation A1, we calculate the chi-square score for each vector in our series. If the observed score is greater than 14.84, that vector is an outlier.

Appendix B. Blended Covariance Matrixes

To identify optimal portfolios based on our view of and attitude toward the two risk regimes, we first augment the standard mean–variance objective function to include the inside covariance matrix Σ_i and the outlying covariance matrix Σ_0, and we assign probabilities to them.[7] The vector of returns has a mean μ and a covariance matrix Σ. We replace the full-sample covariance matrix Σ with

$$p\Sigma_i + (1 - p)\Sigma_0, \tag{B1}$$

where p is the probability of falling within the inside sample and $1 - p$ is the probability of falling within the outlier sample.[8]

Substituting these two covariance matrixes into the standard equation for the expected utility, *EU*, of a portfolio with a weight vector w yields

$$EU = w'\mu - \lambda\,[p w'\Sigma_i w + (1 - p) w'\Sigma_0 w], \tag{B2}$$

where λ equals aversion to full-sample risk.

Equation B2 allows us to express views about the respective probabilities of the two risk regimes, but it assumes we are equally averse to both regimes. To differentiate our aversions to the two regimes, we first assign values that reflect our relative aversion to each. Then, we rescale those values so that they sum to 2. For example, suppose our aversion to inside risk equals 2 and our aversion to outlier risk equals 3. We rescale our inside risk aversion to equal 0.80 and our outlier risk aversion to equal 1.20, as follows:

$$\lambda_i^* = \frac{2\lambda_i}{\lambda_i + \lambda_0} \tag{B3}$$

$$\lambda_0^* = \frac{2\lambda_0}{\lambda_i + \lambda_0}. \tag{B4}$$

We then multiply the probability-weighted inside and outlying covariance matrixes by their respective rescaled risk aversions:

$$EU = w'\mu - \lambda[\lambda_i^* p w'\Sigma_i w + \lambda_0^*(1 - p)\, w'\Sigma_0 w]. \tag{B5}$$

Although Equation B5 has the virtue of transparency, it is somewhat cumbersome. We can simplify it by defining a grand covariance matrix to equal

$$\Sigma^* = \lambda_i^* p\Sigma_i + \lambda_0^*(1 - p)\Sigma_0. \tag{B6}$$

This definition allows us to recast the objective function as

$$EU = w'\mu - \lambda(w'\Sigma^* w), \tag{B7}$$

which is the original Markowitz objective function.

Notes

1. Markowitz was awarded the Nobel Prize in economics in 1990 for his pioneering work in portfolio selection.
2. Many of the small-sample problems, together with suggested solutions, are well documented by Michaud (1998).
3. If we were to consider three return series, the outlier boundary would be an ellipsoid, a form that looks something like a football. We are unable to visualize an outlier boundary for samples that include more than three return series—at least not without the assistance of a controlled substance.
4. Remember that "regime" in this context refers to the inside and outlier samples, not necessarily to a period that is chronologically contiguous.
5. In the limit, the estimator is meaningless if there are fewer observations than assets. For large covariance matrixes, a factor model may be necessary to restrict the number of parameters in the matrix.
6. The expected returns were chosen for their reasonableness and are not related to either the full-sample means or the outlier-sample means, nor do they represent the views of the Harvard Management Company.
7. We modeled asset returns as if they came from a discrete mixture of two different normal distributions. To derive the chi-square test, we assumed that all returns came from the same normal distribution. This minor contradiction is similar to the contradiction that occurs in standard statistics when we use a test that is valid under a null hypothesis to reject that null hypothesis. In research in progress, we are studying a theoretically more efficient outlier test that makes full use of the model in Equation B1. Whether it will differ much from the simple chi-square test is unclear.
8. Note that this equality also applies exactly to the estimates of these quantities as long as the estimation is consistent with the model—that is, the number of observations from the inside and outlier scenarios are T_i and T_0 such that $T_i / T = (T - T_0) / T = p$.

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